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Zero-selectors and GO spaces ☆

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Dedicated to Professor Jan Aarts

Abstract

We are dealing with Vietoris continuous zero-selectors, i.e., they choose for each non-empty closed set F an isolated point in F. We show that the presence of a continuous zero-selector even on a small class of non-empty closed sets of a space X implies that X is scattered if X is metrizable or non-Archimedean or a P-space. Finally, using continuous zero-selectors, we characterize suborderable spaces which are subspaces of ordinals.

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0. Introduction

We continue the study of continuous *zero-selectors*, i.e., selectors, which are continuous with respect to the Vietoris topology on the family of non-empty closed sets of a given space X and which choose a (relatively) isolated point from each non-empty closed set. Clearly, the existence of an arbitrary zero-selector for X implies that X must be scattered.

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In the part 1.1, we are going to show that, under some additional restrictions to the space, X must be scattered even when a continuous zero-selector acts on a small subfamily of closed sets (see Theorem 1.5 below and its corollaries). In the part 1.2, following reasoning from [15], we show that the density of a regular space X with a continuous zero-selector is equal to the cardinality of X.

Continuous zero-selectors can be defined quite easily for ordinals: just take for each non-empty closed set its minimum. In Section 2, we are dealing with the "opposite" problem: which suborderable spaces with a continuous zero-selector are homeomorphic to a subspace of ordinals? We give a characterization of these suborderable spaces in Theorem 2.9 and we present examples that conditions used in Theorem 2.9 cannot be weakened. Let us recall in this connection that continuous zero-selectors were used to characterize compact ordinal spaces in [9], i.e., it was shown in particular that any compact space with a continuous zero-selector is homeomorphic to a space of ordinals. This fact was afterwards generalized for pseudocompact spaces [2].

1. Zero-selectors

All spaces considered in this paper are Hausdorff. Let $\mathfrak{F}(X)$ be the set of all non-empty closed subsets of X, equipped with the Vietoris topology [16,7]. A base for the Vietoris topology on a subspace $\mathfrak{A}$ of $\mathfrak{F}(X)$ consists of all sets of the form:

$$\langle U^0, U^1, \dots, U^n\rangle = \left\{F \in \mathfrak{A}\colon\ F \subseteq \bigcup_{i\leqslant n} U^i \text{ and } F \cap U^i \neq \emptyset \text{ for every } i \leqslant n\right\}$$

where $U^0, \dots, U^n$ are open subsets of X.

A (continuous) selector on a subspace $\mathfrak{A}$ of $\mathfrak{F}(X)$ is a (continuous) map $\sigma : \mathfrak{A} \to X$ such that $\sigma(F) \in F$ for every $F \in \mathfrak{A}$, The selector σ is said to be a zero-selector provided that $\sigma(F)$ is relatively isolated in F for each $F \in \mathfrak{A}$. We say that X has a continuous (zero-)selector if there exists a continuous (zero-)selector on the whole $\mathfrak{F}(X)$.

The set of cluster points of a set E is denoted by E'.

1.1. Zero-selectors on smaller families

We start with some facts which will be useful in the sequel.

Lemma 1.1. *Let σ be a continuous selector on a subspace $\mathfrak{A}$ of $\mathfrak{F}(X)$, $C \in \mathfrak{A}$, $p = \sigma(C)$. Assume $p \in C'$. Then for every neighbourhood W of p there exists a non-empty finite subset $\Gamma(W)$ of $C \setminus \{p\}$ such that $\sigma(G) \in W$ for each $G \in \mathfrak{A}$, with $\Gamma(W) \subseteq G \subseteq C$.*

Proof. By the continuity of σ, there exist non-empty open sets $U^0, U^1, \dots, U^n$ such that $C \in \langle U^0, U^1, \dots, U^n\rangle \subseteq \sigma^{-1}(W)$. Since $p \in C'$, we can choose a point $p_i \in U^i \cap C$, for every $i \leqslant n$. The required set is $\Gamma(W) = \{p_1, \dots, p_n\}$. □

Notice that the condition $p \in C'$ in the above lemma was needed to ensure that $\Gamma(W)$ does not contain p. If p is isolated in C then it is not excluded that $U^i \cap C = \{p\}$ for some i.

Lemma 1.2. *Let X be a regular space and let $\mathfrak{A}$ be a subspace of $\mathfrak{F}(X)$ such that $G \cup F \in \mathfrak{A}$ whenever $G \in \mathfrak{A}$ and F is finite. Let σ be a continuous selector on $\mathfrak{A}$, $C \in \mathfrak{A}$, $p = \sigma(C)$. Assume that p is relatively isolated in C. There exist a neighbourhood V of p, with $\overline{V} \cap C = \{p\}$, and a non-empty finite subset F of $C \setminus \overline{V}$ such that $\sigma(H \cup G) \in G$ for every finite subset H, with $F \subseteq H \subseteq C \setminus \overline{V}$ and for every element G of $\mathfrak{A}$ contained in V.*

Proof. Let W be a neighbourhood of p such that $\overline{W} \cap C = \{p\}$ and let $\langle U^0, U^1, \ldots, U^n \rangle$ be the Vietoris neighbourhood of C contained in $\sigma^{-1}(W)$. It is not restrictive to assume that $p \in U^0 \subseteq W$ and that $U^i \cap \overline{W} = \emptyset$ for every $i > 0$. For each $i > 0$, choose a point $p_i \in U^i \cap C$. By putting $V = U^0$ and $F = \{p_1, \ldots, p_n\}$, we get the conclusion. □

Proposition 1.3. *If X is normal and has a continuous zero-selector, then every subspace of X has a continuous zero-selector.*

Proof. See [2, Proposition 2.10]. □

The space X is said to be scattered if every non-empty (closed) subset F of X has a point relatively isolated in F. Notice that X is scattered if and only if it has a zero-selector, not necessarily continuous.

Let $\mathcal{S}$ denote the family of the countably infinite closed subsets of X with at most one limit point.

Lemma 1.4. *Let X be a dense in itself regular space. If each point has a local base linearly ordered by inclusion, then every non-empty open set contains an element $F \in \mathcal{S}$.*

Proof. Let V be a non-empty open set and let W be a non-empty open set with $\overline{W} \subseteq V$. Take any countably infinite subset $E \subseteq W$. If $E' = \emptyset$, it suffices to choose $F = E$. Otherwise let p be a point of E'. Then $p \in \overline{W} \subseteq V$. Since the local base at p is linearly ordered, the local character at p is countable. Then there exists a sequence $\{x_n\colon x_n \in E\}$ converging to p and one may put $F = \{x_n\} \cup \{p\}$. □

Recall that a space X is called non-Archimedean [17][2] if there exists a base $\mathcal{B}$ such that if B_1 and B_2 are members of $\mathcal{B}$ with $B_1 \cap B_2 \neq \emptyset$, then either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Such a base is said to be a non-Archimedean base. The union of a chain of elements of $\mathcal{B}$ is a clopen set; the intersection is either a singleton or a clopen set. Consequently every point p has a base of clopen neighbourhoods $\{V_\alpha\colon \alpha < \chi_p\}$ well ordered by reversed set inclusion (χ_p denotes the local character at p). Non-Archimedean spaces are strongly zero-dimensional and every open cover is refined by a partition of open sets.

Theorem 1.5. *Let X be a regular space such that there exists a continuous zero-selector σ defined on $\mathcal{S}$. Then X is scattered in the following cases*:

[2] For more recent results on non-Archimedean spaces, the reader should consult [18] and other Nyikos's papers cited there.

(1) *X is a non-Archimedean space.*
(2) *X is a metric space.*
(3) *Every countable subset of X is closed* (*e.g., if X is a P-space*).

Proof. It is enough to show that each closed subset has an isolated point. Further, since every closed subspace satisfies the hypothesis, it is enough to prove that X has an isolated point. By way of contradiction, suppose that every point is a limit point. Then every non-empty open set contains an element of $\mathcal{S}$ (in cases (1) and (2) apply Lemma 1.4, in case (3) observe that any countable subset of X is closed and discrete). Let $V_{-1} = X$ and $G_0 \in \mathcal{S}$. By Lemma 1.2, there exist an open neighbourhood V_0 of $\sigma(G_0)$ and a non-empty finite subset F_0 of $G_0 \setminus \overline{V}_0$ such that $\sigma(F_0 \cup G) \in G$ for every element G of $\mathcal{S}$ contained in V_0. Take any $G_1 \subseteq V_0$, $G_1 \in \mathcal{S}$. Still by Lemma 1.2, there exist an open neighbourhood V_1 of $\sigma(F_0 \cup G_1)$, with $\overline{V}_1 \subseteq V_0$, and a non-empty finite subset F_1 of $G_1 \setminus \overline{V}_1$ such that $\sigma(F_0 \cup F_1 \cup G) \in G$ for every element G of $\mathcal{S}$ contained in V_1. Proceeding by induction, we get non-empty open sets V_k and non-empty finite subsets F_k such that:

(a) $\overline{V}_k \subseteq V_{k-1}$, $F_k \subset V_{k-1} \setminus \overline{V}_k$,
(b) $\sigma(F_0 \cup F_1 \cup \cdots \cup F_k \cup G) \in G$ $\forall G \subseteq V_k$, $G \in \mathcal{S}$.

Put $F = F_0 \cup F_1 \cup \cdots \cup F_k \cup \cdots$ and $H = \overline{F}$. Property (a) implies that F is a discrete subspace of X.

We will show that each of the hypothesis (1)–(3) listed below allows us to slightly modify the construction above in order to obtain $H \in \mathcal{S}$.

(1) Let $\mathcal{B}$ denote a non-Archimedean base of X. Then the neighbourhoods V_k may be chosen in $\mathcal{B}$, and consequently $\bigcap_{k \geqslant 0} V_k$ is either a single point or a (possibly empty) clopen set. Then H contains at most one limit point and consequently it belongs to $\mathcal{S}$.
(2) Take the neighbourhoods V_k in such a way that $\operatorname{diam}(V_k) < 2^{-k}$. Then F is a Cauchy sequence and consequently $H \in \mathcal{S}$.
(3) In this case $H = F$ is a closed discrete subspace, hence $H \in \mathcal{S}$.

For every $k \in \mathbb{N}$, the subset $H_k = H \setminus (\bigcup_{i=0}^{k} F_i)$ belongs to $\mathcal{S}$ and is contained in V_k. Consequently, by (b), we have $\sigma(H) \in H_k$ for every $k \in \mathbb{N}$ and hence $\sigma(H) \in H \setminus F$. Therefore $\sigma(H)$ must be a limit point of H, a contradiction (notice that in cases (1) and (2) σ is a continuous zero-selector, in case (3) $H' = \emptyset$). $\square$

Corollary 1.6. *Let X be a regular space which satisfies one from* (1), (2)*, or* (3) *in Theorem* 1.5. *If there exists a continuous zero-selector on the family of all non-empty scattered closed subset of X, then X is scattered.*

Let $\mathcal{D}$ denote the family consisting of all sets which are closures of some countably infinite discrete subspace.

Theorem 1.7. *Let X be a regular space such that every point has a local base linearly ordered by inclusion. If there exists a continuous zero-selector σ defined on $\mathcal{D}$, then X is scattered.*

Proof. Argue as in the proof of Theorem 1.5, observing that the set H belongs to $\mathcal{D}$. $\square$

The hypothesis of Theorem 1.5 in the case (3) may be weakened by the following extension of [1, Lemma 2.9].

Proposition 1.8. *Assume that every countable subset of a Hausdorff space X is closed. Then every continuous selector σ on $\mathfrak{A} \subseteq \mathfrak{F}(X)$, satisfying $\mathcal{S} \subseteq \mathfrak{A}$, is a continuous zero-selector.*

Proof. By way of contradiction, suppose that there exists a non-empty closed subset C of X such that $\sigma(C) = p \in C'$. With an iterated application of Lemma 1.1, construct a chain $\{W_n\}_{n\in\omega}$ of neighbourhoods of p such that $W_n \supseteq W_{n+1}$ and $W_{n+1} \cap \Gamma(W_n) = \emptyset$ for every n. The subset $G = \bigcup_n \Gamma(W_n)$ is closed and $\Gamma(W_n) \subseteq G \subseteq C \setminus \{p\}$ for every n. Thus $\sigma(G) \in G \cap (\bigcup_n W_n) = \emptyset$, a contradiction. $\square$

1.2. Zero-selectors and cardinal functions

In the paper [15] it is proved that a regular separable space with an uncountable closed discrete subset has no continuous selector. A slight modification of the argument used there leads to the following:

Theorem 1.9. *If X is a regular space with a continuous zero-selector, then the density of X coincides with the cardinality of X.*

Proof. Denote by σ the continuous zero-selector. By transfinite induction, define a well-ordering $X = \{x_\alpha \colon \alpha < \lambda\}$ by letting $x_0 = \sigma(X)$ and $x_\alpha = \sigma(X_\alpha)$, where:

$$X_\alpha = X \setminus \{x_\beta \colon \beta < \alpha\}.$$

Notice that X_α is closed and x_α is isolated in X_α [2, Theorem 2.1]. Fix open neighbourhoods O_α of each x_α such that:

$$\overline{O_\alpha} \cap X_\alpha = \{x_\alpha\}.$$

By the continuity of σ, for each α there are open sets U_α^n, $n < m_\alpha$, such that:

$$X_\alpha \in \langle U_\alpha^n \colon n < m_\alpha \rangle \subseteq \sigma^{-1}(O_\alpha).$$

In particular, $X_\alpha \cap U_\alpha^n \neq \emptyset$ for every $n < m_\alpha$. We may and shall assume that $x_\alpha \in U_\alpha^0$ and $1 < m_\alpha$. Modifying the U_α^n's, we may assume that $\forall \alpha < \lambda$:

(a) $U_\alpha^0 \subseteq O_\alpha$.
(b) $\overline{O_\alpha} \cap \bigcup_{0<n<m_\alpha} U_\alpha^n = \emptyset$.

Really, to achieve (a), just take $U_\alpha^0 \cap O_\alpha$. This may create a problem as there could be points in $X_\alpha \cap (U_\alpha^0 \setminus \{x_\alpha\})$ which are contained just in U_α^0. That is why we add the set $U_\alpha^0 \setminus \overline{O_\alpha}$ to the set U_α^1.

To achieve (b), subtract $\overline{O_\alpha}$ from U_α^n for all n, $0 < n < m_\alpha$.

Consequently:

(c) for all $F \in [X]^{<\omega}$ if $F \in \langle U_\alpha^n\colon\ n < m_\alpha\rangle$, then $\sigma(F) \in F \cap U_\alpha^0$.

By contradiction, assume there exists a dense subset D such that $|D| < |X|$.

Claim. *There are $F \in [D]^{<\omega}$ and $\alpha < \beta < \lambda$ such that*:

(d) $F \in \langle U_\alpha^n\colon\ n < m_\alpha\rangle \cap \langle U_\beta^n\colon\ n < m_\beta\rangle$.
(e) $F \cap U_\alpha^0 \cap U_\beta^0 = \emptyset$.

First note that the claim leads to a contradiction. Namely, by (c), $\sigma(F) \in U_\alpha^0 \cap U_\beta^0$ but by (e) this is impossible. In order to prove the claim, for each α let

$$V_\alpha = \bigcup_{0<n<m_\alpha} U_\alpha^n.$$

Then $U_\alpha^0 \cap V_\alpha = \emptyset$ by (a) and (b).

As $|D| < |X|$, a simple argument, based on the arithmetic of cardinals, shows that there exist a subset $J \subseteq \lambda$, with $|J| > |D|$, and a finite subset $G \subseteq D$ such that:

$$\forall \alpha \in J\ \forall n,\ 0 < n < m_\alpha\colon\ G \cap U_\alpha^n \neq \emptyset \text{ and } G \subseteq V_\alpha.$$

Let $\{\delta_\alpha\colon\ \alpha < \mu\}$ be an increasing enumeration of J. For every $\alpha < \mu$, consider the non-empty set (recall that $x_{\delta_\alpha+1} \in X_{\delta_\alpha}$ and $x_{\delta_\alpha+1} \notin O_{\delta_\alpha}$):

$$D_{\alpha+1} = D \cap U_{\delta_{\alpha+1}}^0 \cap V_{\delta_\alpha}.$$

As $|D| < |J|$, the sets $D_{\alpha+1}$ are not pairwise disjoint. So we may fix ordinals $\alpha < \beta < \mu$ such that:

$$U_{\delta_{\alpha+1}}^0 \cap V_{\delta_\beta} \neq \emptyset, \quad \text{hence } \alpha + 1 < \beta.$$

Let $d_0 \in D \cap U_{\delta_{\alpha+1}}^0 \cap V_{\delta_\beta}$. As $D \cap U_{\delta_\beta}^0 \cap V_{\delta_{\alpha+1}} \neq \emptyset$ (recall again that $x_{\delta_\beta} \in V_{\delta_{\alpha+1}}$), we may choose $d_1 \in D \cap U_{\delta_\beta}^0 \cap V_{\delta_{\alpha+1}}$. Now define $F = G \cup \{d_0, d_1\}$. Notice that $F \cap U_{\delta_{\alpha+1}}^0 = \{d_0\}$ and $F \cap U_{\delta_\beta}^0 = \{d_1\}$. It is clear that F, $\delta_{\alpha+1}$, δ_β satisfy the conclusions of the claim. □

Remark 1.9.1. A selector σ on a subfamily $\mathfrak{A}$ of $\mathfrak{F}(X)$ is said to be H-continuous provided that for every $F \in \mathfrak{A}$ and for every neighbourhood V of $\sigma(F)$ in X, there exists an open neighbourhood W of $\sigma(F)$ in X such that, for every $G \in \mathfrak{A}$, with $G \cap W \neq \emptyset$ and $G \setminus W = F \setminus W$, the point $\sigma(G)$ belongs to V. This condition is weaker than usual Vietoris continuity and every scattered space admits an H-continuous zero-selector on $\mathfrak{F}(X)$ (see the proof of (2) $\Rightarrow$ (3) in [1, Theorem 1.3]). However, Theorem 1.9 implies that there are scattered spaces without any continuous zero-selector—the space Ψ as defined in [10] can serve as an example; one could use results from [9] to obtain the same conclusion.

2. Embedding into ordinals

In the present section we are going to discuss the properties of (scattered) GO spaces which admit a continuous zero-selector, in order to characterize the GO spaces which are subordinal spaces. *Subordinal space* means a space which is embeddable[3] into some ordinal space (with its order topology).

A Hausdorff space X is a GO space (generalized ordered space) if there exists a linear order $<$ on X such that each point has a local base consisting of (possibly degenerate) intervals. Such an order is said to be a *compatible* order for X. Notice that the open interval topology of a compatible order is finer than the original topology. E. Čech proved that the class of GO spaces coincides with the class of suborderable spaces, that is spaces which are embeddable into some linearly ordered spaces [3, Theorem 17A22, 17A23]. GO spaces are monotonically normal, hence hereditarily collectionwise normal [12]. We refer an interested reader, e.g., to [14] for more information.

A space is orderable if it is homeomorphic to some linearly ordered space. A GO space is orderable if it is either scattered [19] or connected or compact. A scattered GO space is strongly zero-dimensional, since it is orderable and hereditarily disconnected [13].

A non-Archimedean space is a GO space (this may be proven by using the fact that there exists a base for the open sets which is a tree by reverse inclusion). This fact was implicitly contained in Kurepa's and Papic's papers from the 1950s (see [20]) and was rediscovered in Nyikos' papers in the 1970s.

Since a topological sum of ordinal spaces is a subordinal space, a carbon copy of the proof of [1, Theorem 1.6] leads to the following result (cf. a more detailed statement in [18, Theorem 2.4]).

Theorem 2.1. *Every non-Archimedean scattered space is a subordinal space.*

We adopt the usual notations for intervals in GO spaces: $(\leftarrow, b]$, $[a, \rightarrow)$, $[a, b), \ldots$.

If p is a point of a GO space X, the left character $\chi^-(p)$ is the character of p in the subspace $(\leftarrow, p]$; the right character $\chi^+(p)$ is defined analogously. The point p has a linearly ordered base of neighbourhoods iff either $\chi^-(p) = \chi^+(p)$ or one among $\chi^-(p)$ and $\chi^+(p)$ is equal to 1. The point p is *two-sided* if both $\chi^-(p)$ and $\chi^+(p)$ are infinite; otherwise it is *one-sided*. These notions depend on the choice of the compatible order on X. When $\chi^-(p)$ is infinite, it coincides with the cofinality at p of the linearly ordered set $(\leftarrow, p)$ (analogously for $\chi^+(p)$). We will always assume that any given GO space is endowed with a compatible linear order.

[3] For which there exists a homeomorphic embedding.

Proposition 2.2. *Let X be a GO space with a continuous zero-selector and let p be a two-sided point. Then $\chi^-(p) = \chi^+(p)$ and consequently every point has a linearly ordered base of neighbourhoods.*

Proof. Let $\lambda = \chi^-(p)$ and $\mu = \chi^+(p)$. By contradiction, suppose $\lambda < \mu$. Choose strictly monotone transfinite sequences $\{x_\alpha\}_{\alpha<\lambda}$ and $\{y_\beta\}_{\beta<\mu}$ converging to p from the left and from the right, respectively.

Suppose there exists $\bar{\alpha} < \lambda$ such that $\sigma([x_{\bar{\alpha}}, y_\beta]) \geqslant p$ for a cofinal set of points y_β. Since the sequence of sets $[x_{\bar{\alpha}}, y_\beta]$ converges to $[x_{\bar{\alpha}}, p]$ in the Vietoris topology, $\sigma([x_{\bar{\alpha}}, p])$ would be equal to p and σ would not be a continuous zero-selector. Consequently, for every $\alpha < \lambda$, there exists $\beta_\alpha < \mu$ such that $\sigma([x_\alpha, y_\beta] \leqslant p, \forall \beta \geqslant \beta_\alpha$. We have $\bar{\beta} = \sup_\alpha \beta_\alpha < \mu$ since μ is a regular cardinal. Then $\sigma([x_\alpha, y_{\bar{\beta}}]) \leqslant p$ for every α, and passing to the limit still leads to a contradiction with the property of σ. □

The rest of this section is almost entirely devoted to the proof of the next theorem. Notice that a subspace of a (scattered) GO space is a (scattered) GO space.

Theorem 2.3. *If there exists a continuous zero-selector on the GO space X, then the set*

$$B = \{x \in X\colon\ x \text{ is two-sided}\}$$

is paracompact.

In the proof we are going to use the following result of Engelking and Lutzer [8, Theorem 2.3].

Theorem 2.4. *A GO space is not paracompact iff it contains a closed set F which is homeomorphic to a stationary subset S of some regular uncountable cardinal κ.*

The reasoning used in their proof will be useful as well. In particular, one sees that the homeomorphism above between F and S could be taken as a monotone map (in the original order of the GO space). Lemma 2.5 below presents a direct proof of this property, using just Fodor's lemma on stationary sets.

It is convenient to introduce the following notation:

If a and b are distinct points of a linearly ordered set, the symbol $I(a,b)$ denotes the interval $(\min\{a,b\}, \max\{a,b\})$.

Lemma 2.5. *Suppose $(Z,<)$ is a GO space containing a subspace F which is homeomorphic to a stationary subset S of some regular uncountable cardinal κ. Denote this homeomorphism by $e: S \to F$. There is a set $T \subseteq S$ which is stationary in κ and e restricted to T is monotone. The set T may be chosen closed in S, so that $e(T)$ is closed if F is closed.*

Proof. Let us observe first the following:

Fact. The set

$$M = \{x \in S\colon \exists y_x,\ z_x \in S \text{ with } y_x < x < z_x,\ e(y_x) \in I(e(x), e(z_x))\}$$

is not stationary.

Proof of Fact. Suppose M is stationary. Then there is a stationary set $P \subseteq M$ such that y_t is constant for $t \in P$. Denote this constant by y_0. Without loss of generality, we may assume that $e(t) < e(z_t)$ for each $t \in P$. The set $Q = \{z_t\colon t \in T\}$ is cofinal in S. Hence the intersection of closures $\overline{P}^S \cap \overline{Q}^S$ is uncountable. But images $e(P)$ and $e(Q)$ are separated by $e(y_0)$—a contradiction. □

Since M is not stationary, there is an unbounded closed set K in κ such that $K \cap M = \emptyset$. Put $D = S \cap K$. Further we distinguish

$$D^+ = \{x \in D\colon e(x) < e((x, \to) \cap D)\},$$
$$D^- = \{x \in D\colon e(x) > e((x, \to) \cap D)\}.$$

It follows from the fact above and the choice of K that $D = D^+ \cup D^-$. Notice that e is clearly increasing on D^+ and decreasing on D^-. If $x \in D^+$ and $y \in D^-$, it follows from the definition of D^+ and D^- that $e(x) < e(y)$, thus $e(D^+) < e(d^-)$.

It means that $\overline{e(D^+)}^F \cap \overline{e(D^-)}^F$ contains at most one point. Hence only one of sets D^+, D^- can be cofinal in D. Assume, e.g., that D^+ is cofinal in D. Then put $T = D^+ \setminus [0, \sup D^-]$. □

The proof of Theorem 2.3 follows now from Theorem 2.4 and the next lemma.

Lemma 2.6. *Suppose that the GO space X has a subspace F consisting of two-sided points such that F is homeomorphic to a stationary subset S of some regular uncountable cardinal κ. Then X does not have any continuous zero-selector.*

Proof. By Lemma 2.5 we may assume that the homeomorphism $\lambda : S \to F$ is a monotone map. Without loss of generality, suppose that λ is increasing and that F is cofinal in X (apply Proposition 1.3, possibly replacing X with $\bigcup_{x\in F}(\leftarrow, x]$); hence the cofinality of X is κ.

Arguing by contradiction, let $\sigma : \mathfrak{F}(X) \to X$ be a continuous zero-selector. If $\alpha \in S$, define

$$t_\alpha = \sigma([\lambda(\alpha), +\infty)).$$

As $\lambda(\alpha)$ is two-sided, we obtain $\lambda(\alpha) < t_\alpha$. Put $M_\alpha = [\lambda(\alpha), +\infty)$. So $t_\alpha \in \mathrm{Isol}(M_\alpha) \subseteq \mathrm{Isol}(X)$.

There is a neighbourhood V_α of M_α and a finite D_α, $t_\alpha \in D_\alpha \subseteq M_\alpha$, such that for each $G \in \mathfrak{F}(X)$,

$$\sigma(G) = t_\alpha \quad \text{whenever } D_\alpha \subseteq G \subseteq V_\alpha. \tag{$*$}$$

Let S' denote the cub of κ consisting of the limit points of S and consider the stationary set $R = S \cap S'$.

So for each $\alpha \in R$, there is $m_\alpha < \alpha$, $m_\alpha \in S$ and $(\lambda(m_\alpha), \rightarrow) \subseteq V_\alpha$ (recall that $\lambda(\alpha)$ is two-sided). There is a stationary set $T \subseteq R$ such that $m_\alpha = m$ constantly for $\alpha \in T$.

Observe that $\lambda(m) < \lambda(\alpha)$ for every $\alpha \in T$, and so $D_\beta \subseteq V_\alpha$ for every $\alpha, \beta \in T$.

Take two distinct elements α, β of T such that $\max(D_\alpha) < \lambda(\beta)$. Then $D_\alpha \cap D_\beta = \emptyset$. Put $Z = D_\alpha \cup D_\beta$. Since $Z \subseteq V_\alpha \cap V_\beta$, by $(*)$, $\sigma(Z) \in D_\alpha \cap D_\beta$, a contradiction. □

Let φ denote an homeomorphic embedding of the space X into some ordinal α. Then X is a GO space and the map $\sigma(F) = \varphi^{-1}(\min \varphi(F))$ is a continuous zero-selector for X. Consequently, the set B of two-sided points for a compatible order on X is necessarily paracompact. However the paracompactness of B does not ensure that a scattered GO space is a subordinal space: e.g., consider the space obtained from the disjoint union of $\omega_1 + 1$ and $(\omega + 1)^*$ by identifying the point $\{\omega_1\}$ and $\{\omega\}$.[4] This scattered space is a LOTS with a single two-sided point and has no continuous zero-selector by Proposition 2.2.

We are now going to prove that the paracompactness of B is a sufficient condition for GO spaces which are locally subordinal spaces.

An one-sided point p is said to be *left-sided* if $\chi^-(p)$ is infinite (analogously for *right-sided*).

Lemma 2.7. *Let C be a GO space with a minimum element a and consisting of one-sided points only. Suppose that, for every $x \in C$, the interval $[a, x]$ is a subordinal space. Then C is a subordinal space.*

Proof. If C has a maximum element, the conclusion is obvious. Otherwise, by transfinite induction, construct an increasing cofinal sequence $\{x_\alpha\}_{\alpha<\lambda}$ and put $E = \{x_\alpha\colon \alpha < \lambda\}$. Notice that a point p belongs to E' if and only if p is a non-isolated left-sided point and there exists a limit ordinal $\gamma < \lambda$ such that $p = \sup\{x_\alpha\colon \alpha < \gamma\}$.

For each $\beta < \lambda$ consider the convex set:

$$\Delta_\beta = \overline{[a, x_\beta)} \setminus \overline{\bigcup_{\alpha<\beta} [a, x_\alpha)}.$$

Since every point is one-sided, the disjoint family $\{\Delta_\beta\}_{\beta<\lambda}$ consists of clopen sets, some of which may be empty. Then $p \in C \setminus \bigcup_{\beta<\lambda} \Delta_\beta$ if and only if $p \in E'$. By hypothesis, for each β there exists an embedding φ_β from the clopen set Δ_β to some ordinal τ_β. The required embedding is the map $\varphi : C \to \sum_{\beta<\lambda} \tau_\beta$ defined as follows:

$$\varphi(x) = \begin{cases} \sum_{\alpha<\beta} \tau_\alpha + 1 + \varphi_\beta(x) & \text{if } x \in \Delta_\beta, \\ \sum_{\alpha<\beta} \tau_\alpha & \text{if } x \in E' \;\&\; x = \sup\{x_\alpha\}_{\alpha<\beta}. \end{cases} \qquad \square$$

Theorem 2.8. *Let X be a GO space which is locally a subordinal space. If the set B of two-sided points is paracompact, then the whole space X is a subordinal space.*

Proof. The hypothesis of local embeddability ensures that X is a zero-dimensional scattered GO space, hence it is strongly zero-dimensional. In [24] Telgársky proved that a

[4] Recall a common notation: for an ordered set L, L^* denotes the reversed ordering on the same underlying set.

paracompact scattered space is ultraparacompact, i.e., every open cover is refined by a disjoint cover of open sets. Recall that each open subset of a GO space is a disjoint union of open convex subsets. So there is a collection $\mathcal{W}_B$ of open convex subsets of X such that

(1) $B \subseteq \bigcup \mathcal{W}_B$;
(2) each $W \in \mathcal{W}_B$ is a subordinal space;
(3) $\mathcal{W}_B$ is a disjoint collection.

If $\mathcal{W}_B$ is a cover of X, the conclusion follows since X is a topological sum of subordinal spaces.

Otherwise, assume that $\bigcup \mathcal{W}_B \neq X$. For each $x \in X \setminus \bigcup \mathcal{W}_B$, we take a convex clopen set C_x such that x is an extreme point of C_x and moreover C_x is a subordinal space. Put $\mathcal{W}_E = \{C_x\colon x \in X \setminus \bigcup \mathcal{W}_B\}$ and $\mathcal{W} = \mathcal{W}_B \cup \mathcal{W}_E$. It is convenient now to follow reasoning from [7, 5.2.22, p. 429]. So $\mathcal{W}$ decomposes into components $\{\mathcal{W}_\iota\colon \iota \in I\}$ of connected families as defined in [7, Lemma 5.3.8]. Unions of these $\mathcal{W}_\iota$ form a disjoint open cover, so it suffices to prove that each $\bigcup \mathcal{W}_\iota$ is a subordinal space.

Fix some $\iota \in I$ and put $Z = \bigcup \mathcal{W}_\iota$ (notice that Z is convex). As in [7, 5.2.22(a)], we deal only with countably (or finitely) many points. There are sequences (finite or infinite, so possibly of different length)

$$x_i, y_i, \quad i = 1, 2, 3, \dots, \text{ of points from } Z$$

such that

(1) $x_{i+1} > x_i$, $y_{i+1} < y_i$, $i = 1, 2, 3, \dots$, $x_1 > y_1$,
(2) $\mathrm{St}(x_{i+1}, \mathcal{W}_\iota) \cap \mathrm{St}(x_i, \mathcal{W}_\iota) \neq \emptyset \neq \mathrm{St}(y_{i+1}, \mathcal{W}_\iota) \cap \mathrm{St}(y_i, \mathcal{W}_\iota)$, $i = 1, 2, 3, \dots$ and moreover $\mathrm{St}(x_1, \mathcal{W}_\iota) \cap \mathrm{St}(y_1, \mathcal{W}_\iota) \neq \emptyset$,
(3) $x_{i+1} \notin \mathrm{St}(x_i, \mathcal{W}_\iota)$, $y_{i+1} \notin \mathrm{St}(y_i, \mathcal{W}_\iota)$, $i = 1, 2, 3, \dots$ and moreover $y_1 \notin \mathrm{St}(x_1, \mathcal{W}_\iota)$,
(4) $Z = \bigcup \{\mathrm{St}(x_i, \mathcal{W}_\iota)\colon i = 1, 2, 3, \dots\} \cup \bigcup \{\mathrm{St}(y_i, \mathcal{W}_\iota)\colon i = 1, 2, 3, \dots\}$.

Let us meditate first on the sequence

$$\cdots < y_2 < y_1 < x_1 < x_2 < \cdots. \tag{$***$}$$

Let us take two neighboring points $a < b$ from this sequence. By (2), there are sets $W_a \ni a$, $W_b \ni b$ from $\mathcal{W}_\iota$ with $W_a \cap W_b \neq \emptyset$. Recall that $W_a \cap W_b$ is open and convex. Choose $\beta_{a,b} \in \mathrm{Isol}(W_a \cap W_b)$. Let us consider three consecutive elements $a < b < c$ from $(***)$. It can be seen easily[5] that the clopen interval $(\beta_{a,b}, \beta_{b,c}]$ is a subordinal space. The proof is concluded if both sides of the sequence $(***)$ are infinite since Z splits into the union of a clopen partition of subordinal spaces. So the problem occurs when $(***)$ is finite on the left or on the right.

As the procedure is quite symmetric, we suppose that $\{x_i\}$ is finite and $\{y_i\}$ is infinite. Consider x_k with the maximal index k. Then $\mathrm{St}(x_k, \mathcal{W}_\iota)$ contains the convex set $[x_k, \rightarrow) \cap Z$.

[5] In a scattered GO space, the union of two convex open sets which are subordinal spaces is a subordinal space.

Put $H = [x_k, \to) \cap Z$. There is at most one $W \in \mathcal{W}_B$ containing x_k such that $W \cap H \cap B \neq \emptyset$. Let p denote the neighboring point of x_k on the left side. The following possibilities may occur on the right side in $(***)$:

(a) *H has a maximum q*:
since there is an element of $\mathcal{W}_\iota$ containing x_k and q, the partition of Z into open subordinal spaces is completed by $(\beta_{p,x_k}, q]$ (by footnote 4).

(b) *W contains H*:
still by footnote 4, the convex open set $(\beta_{p,x_k}, x_k) \cup W$ is a subordinal space which completes the partition of Z.

(c) *None of above cases*:
take an isolated point $r \in H \setminus \{x_k\}$ with $r > W$. The clopen interval (β_{a,x_k}, r) is a subordinal space. By Lemma 2.7, the open set $C = [r, \to) \cap H$ is the subordinal space which completes the partition of Z.

When the binary sequence $(***)$ is finite, the proof may be carried out with similar arguments. □

Theorems 2.3 and 2.8 may be put together to obtain the following characterization.

Theorem 2.9. *Let X be a GO space which is locally a subordinal space and let B be the set of two-sided points. The following conditions are equivalent*:

(i) *X is a subordinal space.*
(ii) *There exists a continuous zero-selector.*
(iii) *B is paracompact.*
(iv) *If Y is a LOTS and $e : X \to Y$ is a homeomorphic embedding, then the set of two-sided points of $e(X)$ is paracompact.*

Proof. (1) ⇒ (2). If φ is the embedding of X into an ordinal space, the map $\sigma(F) = \varphi^1(\min \varphi(F))$ is a continuous zero-selector.

(2) ⇒ (3). By Theorem 2.3.

(3) ⇒ (1). By Theorem 2.8.

(2) ⇒ (4). The existence of continuous zero-selectors is preserved by homeomorphisms. Then $e(X)$ has a continuous zero-selector and Theorem 2.3 leads to the conclusion.

(4) ⇒ (1). By Theorem 2.8, $e(X)$ is a subordinal space. □

A LOTS which is locally a subordinal space may fail to be a subordinal space.

Example 2.10. Consider the following lexicographically ordered space:

$$X = \{(\alpha, 0)\colon\ \alpha < \omega_1\} \cup \left\{\left(\alpha, \frac{1}{n}\right)\colon\ \alpha \in \mathrm{Lim}(\omega_1),\ n = 1, 2, \ldots\right\}.$$

This space is locally homeomorphic to $\omega + 1$, but $B = \{(\alpha, 0)\colon\ \alpha \in \mathrm{Lim}(\omega_1)\}$ fails to be paracompact.

Proposition 2.11. *If X is a countable scattered GO space, then X is a completely metrizable subordinal space.*

Proof. A countable GO space is Lindelöf and satisfies the first countability axiom. The conclusion follows from [24, Theorems 8 and 9]. □

Corollary 2.12. *Assume that the GO space X has a continuous zero-selector. Any of the following conditions imply that X is a subordinal space*:

(i) *X is locally countable.*
(ii) *X is locally pseudocompact.*

Proof. (i) Follows from 2.11 and 2.9; (ii) Follows from 2.9 and [2, Theorem 2.9]. □

Example 2.13. A GO space X with a continuous zero-selector and a single two-sided point does not necessarily admit a compatible order for which every point is one-sided. In particular, X is not subordinal.

Proof. Let $A = [0, \omega_1)$ with the usual order and $B = [0, \omega_1)^*$ with this order reversed. Put $X = A \cup \{\omega_1\} \cup B$, ordered in such a way that $A < \omega_1 < B$ and refine the order topology of X by declaring isolated all points of B. Obviously X is a GO space.[6] Suppose that there exists a compatible order $\prec$ on X in such a way that every point is one-sided and assume that ω_1 is left-sided. Observe that $X \setminus \{y\colon\ x \prec y \prec \omega_1\}$ is countable for each $x \prec \omega_1$. Then the sets A and B are placed on the left of ω_1, except for a countable set of points. Choose sequences $a_1 \prec b_1 \prec \cdots \prec a_i \prec b_i \prec \cdots \prec \omega_1$, with $a_i \in A$ and $b_i \in B$ for each i. Since the countably infinite subset $\{a_i\}$ of A has a limit point $q \in A$, we obtain that q must be left-sided, and consequently q is a limit point also for B, a contradiction.

Now we construct a continuous zero-selector for X. If F denotes a non-empty closed subset of X, put $a = \inf F \cap (A \cup \{\omega_1\})$ and $b = \sup F \cap (\{\omega_1\} \cup B)$. Put:

$$\sigma(F) = \begin{cases} b & \text{if } b \text{ is less than } a \text{ as ordinal numbers,} \\ a & \text{otherwise.} \end{cases}$$

It is straightforward to check that σ is a continuous zero-selector. □

3. An open problem

A construction, under the Diamond Principle, of a locally compact locally countable (abbreviated as LCC) *non-normal* topological space of the cardinality ω_1 with a continuous zero-selector was presented in [2]. This construction produced a *ladder* space; it is clear that each ladder space, more generally each topological space which is a continuous 1–1 preimage of an ordinal, has a continuous zero-selector. One should recall results

[6] As observed by the referee, replacing B with the lexicographic product $B \times \mathbb{Z}$, one sees that X is actually LOTS.

on the Weak Diamond Principle and uniformization and related topological constructions of ladder spaces from [4,5], [22, Appendix], [6, Theorem 18] to find out that for ω_1, the existence of a ladder space which would serve as an example with properties mentioned above, is equivalent with $2^{\aleph_0} < 2^{\aleph_1}$. The Weak Diamond Principle holds true for some cardinals in ZFC [22, Appendix] however we cannot perform our construction as the Weak Diamond Principle need not be true when restricted to ordinals of countable cofinality [21]. Nevertheless, it seems very likely that [23] gives the existence of cardinals in ZFC which allow this construction. We will return to it in a forthcoming paper.

The paper [11] is related to the investigation of topological spaces with a continuous zero-selector as well. However, it is proved there that under MA + nonCH, all continuous bijective preimages of ω_1 are normal, provided they are locally compact. It could indicate that the existence of an LCC non-normal topological space of the cardinality ω_1 with a continuous zero-selector even outside ladder spaces might depend on axioms of the set-theory. Nevertheless, it is connected with another, still open problem:

Problem 3.1. Let X be a topological space with a continuous zero-selector. Does there exist a continuous 1–1 map $f : X \to \kappa$ for some ordinal κ?

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